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KASR TARTIBLI UMUMLASHGAN RIMAN LIUVILL DIFFERENSIAL OPERATORLI TENGLAMA UCHUN TESKARI MASALA

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Annotatsiya. *Ushbu maqolada boshlang‘ich-nolokal chegaraviy shartli vaqt kasr to‘lqin tenglamasi uchun teskari masala o‘rganildi. Biz boshlang‘ich-nolokal chegaraviy shartli masalani tadqiq etishda, dastlab fazo o‘zgaruvchisiga bog‘liq spektral masala o‘rganildi. Spektral masalaning xos soni va xos funksiyalari aniqlandi. Vaqt o‘zgaruvchisi bo‘yicha Koshi masalasi olindi. Bu Koshi masalasi ekvivalent bo‘lgan integral tenglama olindi. Integral tenglamaning yechimi mavjudligi va yagonaligi isbotlandi. So‘ngra boshlang‘ich chegaraviy masala yechimi qator ko‘rinishda izlaymiz. Qatorning tekis darajada uzluksizligi isbotlaymiz. Shundan so‘ng kasr tartibli umumlashgan Riman Liuvill differensial operatorli tenglama uchun teskari masala" tadqiq etamiz, unda to‘g‘ri masala yechimiga qo‘shimcha shart berish orqali tenglamada qatnashuvchi nomalum koeffitsiyentni aniqlash masalasi o‘rganamiz. Bunda asosan teskari masala unga ekvivalent integral tenglamaga keltiramiz, so‘ngra siqiluvchan akslantirishlar prinsipi yordamida teskari masalaning yechimi mavjudligi va yagonaligi isbotlaymiz.*

Tayanch so‘zlar: *Hilfer kasr hosilasi, boshlang‘ich chegaravir shart, Koshi masalasi, integral tenglama, mavjudlik, yagona , tekis yaqinlashuvchi qator.*

$\Omega = \{(x,t): 0 < x < l,\ 0 < t \le T\}$ sohada quyidagi kasr to‘lqin tenglamasini ko‘rib chiqamiz

$$\left(D_{0+,t}^{\alpha,\beta} u\right)(x,t) - u_{xx} + q(t)u(x,t) = f(x,t), \tag{1}$$

Koshi tipidagi boshlang‘ich shartlar bilan

$$I_{0+,t}^{(1-\alpha)(1-\beta)} u(x,t)|_{t=0} = \varphi(x), \tag{2}$$

va chegara shartlari

$$u(0,t)=u(l,t),\ u(0,t)=\int_0^1 u(x,t)\,dx,\quad 0\le t\le T. \qquad (3)$$

Bu yerda $D_{0+,t}^{\alpha,\beta}-$ operator tartibli $0<\alpha<1$ tartibli va $0\le\beta\le1$ tipli umumlashgan Riman-Liouville (Hilfer) kasr differentsial operatori bo‘lib quyidagicha aniqlanadi **[1, pp. 112-118], [2, pp. 62-65]:**

$$D_{0+,t}^{\alpha,\beta}u(\cdot,t))=\left(I_{0+,t}^{\beta(1-\alpha)}\frac{\partial^2}{\partial t^2}\left(I_{0+,t}^{(1-\beta)(1-\alpha)}u\right)\right)(\cdot,t),$$

$$I_{0+,t}^{\gamma}u(x,t)=\frac{1}{\Gamma(\gamma)}\int_0^t\frac{u(x,\tau)}{(t-\tau)^{1-\gamma}}d\tau,\qquad \gamma\in(0,1)$$

$u(x,t)$ funksiyaning t o‘zgaruvchiga nisbatan Riman-Liuvil kasr integrali **[1, s. 112-118], [2, s. 62-65]**, $\Gamma(\cdot)-$ Eylerning Gamma funksiyasi. $f(x,t)$, $\varphi(x)$, funksiyalar berilgan yetarlicha silliq funksiyalar.

$f(x,t)$, $\varphi(x)$, $q(t)$ berilgan funksiyalar va $\alpha\in(0,1)$, $\beta\in[0,1]$ sonlar uchun (1)-(3) boshlang‘ich chegaraviy masala yechimini topish masalasi to‘g‘ri masala deb ataladi.

Faraz qilaylik, ushbu maqola davomida berilgan $\varphi(x), f(x,t)$ funksiyalar quyidagi shartlar o‘rinli bo‘lsin:

A1) $\varphi(x)\in C^3[0,1]$, $\varphi^{(4)}\in L_2[0,1]$, $\varphi(0)=\varphi(1)=0$, $\varphi''(0)=\varphi''(1)=0$, $i=1,2$;

A2) $f(x,\cdot)\in C[0,T]$ va $t\in[0,T]$, $f(\cdot,t)\in C^3[0,1]$, $f(\cdot,t)^{(4)}\in L_2[0,1]$, $f(0,t)=f(1,t)=0$, $f_{xx}(0,t)=f_{xx}(1,t)=0$.

Keyingi b**o‘**limda biz ba’zi kerakli dastlabki ma’lumotlarni taqdim etamiz.

Dastlabki tushunchalar

Ikki parametrli Mittag-Leffler funksiyasi $E_{\alpha,\beta}(z)$ quyidagi qatorlar orqali aniqlanadi:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

bu yerda $\alpha,\beta,z\in\mathbb{C}$ va $\Re(\alpha)>0$, bunda $\Re(\alpha)-$ α kompleks sonining haqiqiy qismi hisoblanadi. Mittag-Leffler funksiyasi ko‘plab mualliflar tomonidan o‘rganilgan va ular turli umumlashmalar va qo‘llanmalar taklif qilishgan va o‘rganishgan.

Biz uzluksiz funksiyalarning og‘irlikli fazolarini ko‘rib chiqamiz:

$$C_\gamma[a,b]:=\{g:(a,b)\to R:(t-a)^\gamma g(t)\in C[a,b],\ 0\le\gamma<1,\},$$

$$C_\gamma^{2,\alpha,\beta}(\Omega)=\{u(x,t):u(\cdot,t)\in C^2(0,1),\ t\in\ [0,T]$$

$$va\ D_{0+,t}^{\alpha,\beta}u(x,\cdot)\in\ C_\gamma(0,T],\ x\in[0,1],\ 1<\alpha\le 2,\ 0\le\beta\le 1\Big\},$$

1-Lemma. [19, 189b] Faraz qilaylik, b $b\ge 0,\ \alpha>0$ va $a(t)$ nol bo‘lmagan, mahalliy integrallanuvchi funksiya bo‘lib, $0\le t<T$ (ba’zi $T\le+\infty$ bo‘lsin. Shuningdek, $u(t)$ ham nol bo‘lmagan va $0\le t<T$ da mahalliy integrallanuvchi bo‘lib,

$$u(t)\le a(t)+b\int_0^t(t-s)^{\alpha-1}u(s)ds,$$

ni qanoatlantirsa, u holda:

$$u(t)\le a(t)+b\Gamma(\alpha)\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(b\Gamma(\alpha)(t-s)^\alpha)a(s)ds.$$

To‘g‘ri masalaning yechimi uchun mavjudlik va yagonalik natijalari

Avvalo, o‘z-o‘ziga qo‘shma bo‘lmagan operator uchun quyidagi tenglama

$$X''(x)+\lambda^2X(x)=0 \tag{5}$$

va quyidagi shartlar bilan

$$X(0)=X(l),\ X(0)=\int_0^1X(x)\,dx. \tag{6}$$

masala qaraylik. Xos qiymat va xos funksiyalar uchun (5)-(6) masalaning yechimi quyidagi shaklda keltirilgan:

$$X_n(x)=c_1\sin(\lambda_nx)+c_2\cos(\lambda_nx), \tag{7}$$

Bu yerda c_1 va c_2 doimiy sonlar, λ_n- esa quyidagi transsendent tenglamaning yechimi:

$$\lambda_n=2tg\frac{\lambda_nl}{2}. \tag{8}$$

Furye usulini qo‘llash orqali (1)-(3) masalaning $u(x,t)$ yechimini shaklning xos funksiyalari bo‘yicha tekis yaqinlashuvchi qatorga yoyamiz:

$$u(x,t)=\sum_{n=0}^{\infty}u_n(t)X_n(x). \tag{9}$$

$n\ge 1$ uchun $u_n(t)$ koeffitsientlarini xos funksiyalarning ortogonalligidan foydalanib topamiz. Ya’ni, (1) ni (7) ning xos funksiyalariga ko‘paytiramiz va $(0,l)$ oraliqda integrallaymiz. $L_2[0,l]$ dagi skalyar ko‘paytma $(f,g)=\int_0^lf(x)g(x)dx$. bilan aniqlanadi. $n\ge 1$ uchun mos ravishda (7) ning xos funksiyalarida $f(x,t)$ va $\varphi(x)$ funksiyalarning Fur'e koeffitsiyentlarini yozamiz:

$$f_n(t)=\int_0^l f(x,t)X_n(x)dx, \qquad \varphi_n=\int_0^l \varphi(x)X_n(x)dx. \qquad (10)$$

In view of (1) for $\left(u(x,t),Y_0(x)\right)=u_0(t)$, we obtain the Cauchy type problem

$$\left(D_{0+,t}^{\alpha,\beta}u_n\right)(t)+q(t)u_n(t)=f_n(t), \qquad (11)$$

$$I_{0+,t}^{(2-\alpha)(1-\beta)}u_n(t)|_{t=0}=\varphi_n. \qquad (12)$$

Biz (11)-(12) masalalarni hal qilamiz.

(11)-(12) boshlang'ich masala $C_\gamma^{\alpha,\beta}[0,T]$ fazosida ikkinchi tur Volterra integral tenglamasiga ekvivalentdir:

$$u_n(t)=\frac{t^{(\beta-1)(1-\alpha)}}{\Gamma\left(1+(\beta-1)(1-\alpha)\right)}\varphi_n+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f_n(\tau)d\tau$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}q(\tau)u_n(\tau)d\tau. \qquad (13)$$

$u_n(t)$ integral tenglamaning yechimi mavjudligi [A] ishda isbotlangan.

(9) tenglikni t o'zgaruvchi bo'yicha differensiallab quyidagi qatorni tuzamiz

$$\left(D_{0+,t}^{\alpha,\beta}u\right)(x,t)=\sum_{n=1}^{\infty}\left(D_{0+,t}^{\alpha,\beta}u_n\right)(t)X_n(x), \qquad (14)$$

$$u_{xx}(x,t)=-\sum_{n=1}^{\infty}\lambda_n^2\,u_n(t)X_n(x). \qquad (15)$$

$\overline{\Omega}:=\{(x,t):0\le x\le 1,\ 0\le t\le T\}$ sohadagi (9), (14) va (15) qatorlarning tekis yaqinlashuvchanligini isbotlaylik. Bu qator har qanday $(x,t)\in\overline{\Omega}$ uchun majaranta qator tuzamiz. Majaranta qatorlar yaqinlashuvchi bo'lishi uchun quyidagi qatorning yaqinlashuvchilikga tekshirish yetarli

$$\sum_{n=1}^{\infty}\lambda_n^2\left(|\varphi_n|+\| f_n \|_{C_\gamma[0,T]}\right). \qquad (16)$$

(16) qatorning yaqinlashishini tekshirishda yordamchi lemmani kiritamiz:

2-Lemma. Faraz qilaylik, A1), A2) shartlar bajarilsin, u holda

$$\varphi_{n,i}=\frac{1}{\lambda_n^4}\varphi_{n,i}^{(4)},\ i=1,2,\ \ f_n=\frac{1}{\lambda_n^4}f_n^{(4)}, \tag{17}$$

bu yerda

$$\varphi_{n,i}^{(4)}=\int_0^1\varphi_i^{(4)}(x)Y_n(x)dx,\ i=1,2, \tag{18}$$

$$f_n^{(4)}=\int_0^1 f^{(4)}(x)Y_n(x)dx, \tag{19}$$

hamda bular uchun quyidagi tengsizliklar o'rinli:

$$\sum_{n=1}^{\infty}\left|\varphi_{n,1}^{(4)}\right|^2\leq\|\varphi_1^{(4)}\|_{L_2[0,l]}, \tag{20}$$

$$\sum_{n=1}^{\infty}\left|\varphi_{n,2}^{(4)}\right|^2\leq\|\varphi_2^{(4)}\|_{L_2[0,l]}, \tag{21}$$

$$\sum_{n=1}^{\infty}\left|f_n^{(4)}\right|^2\leq\|f^{(4)}\|_{L_2[0,T]}. \tag{22}$$

Isbot. 3-lemma shartlarida (17) tenglik $\varphi(x),\ \psi(x)$ va $f(x,t)$ funksiyalarning Furye koeffitsientlari uchun integrallarda qismlarga ikki marta integrallash orqali osonlik bilan olinadi. (20)-(22) munosabatlar $\varphi''(x),\ \psi''(x)$ va $f_{xx}(x,t)$ funksiyalari uchun Bessel tengsizliklaridir.

E'tibor bering, 2-lemma shartlarida v qatorlar yaqinlashadi. 20)-(22) qatorlarga va Koshi-Bunyakovskiy tengsizligini qo'llasak, quyidagini olamiz

$$\sum_{n=1}^{\infty}\lambda_n^2\left(|\varphi_n|+\|f_n\|_{C_\gamma[0,T]}\right)=\sum_{n=1}^{\infty}\left(\frac{\varphi_n^{(4)}}{\lambda_n^2}+\frac{\|f_n^{(4)}\|_{C_\gamma[0,T]}}{\lambda_n^2}\right)$$

$$\leq\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\left|\varphi_n^{(4)}\right|^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\left\|f_n^{(4)}\right\|^2_{C_\gamma[0,T]}\right)^{\frac{1}{2}}$$

$$\leq C\left(\|\varphi_2^{(4)}\|_{L_2[0,l]}+\|f^{(4)}\|_{L_2[0,l]}\right).$$

Yuqoridagi tengsizlikdan (16) qatorning yaqinlashuvchanligi kelib chiqadi.

Bundan esa (9), (14) va (15) qatorlarning tekis yaqinlashuvchanligi kelib chiqadi. Shunday qilib, biz quyidagi teoremani isbotladik.

1-Teorem. Faraz qilaylik, $q(t) \in C[0,T]$ bo‘lib, A1), A2) shartlar bajarilsin, u holda (1)-(1) to‘g‘ri masalaning $u(x,t) \in C_{\gamma}^{2,\alpha,\beta}(\overline{\Omega})$ sinfga tegishli yagona yechimi mavjud bo‘ladi.

Endi kasr tartibli umumlashgan Riman Liuvill differensial operatorli tenglama uchun teskari masala

Teskari masalada (1)-(3) masalaning yechimi quyidagi qo‘shimcha shartni qanoatlantirsa, (1) tenglamadan $q(t),\ t \in [0,T]$ funksiyani aniqlash talab qilinadi:

$$\int_0^1 w(x)u(x,t)dx = h(t), \quad 0 \le t \le T, \tag{23}$$

bu yerda $w(x)$ va $h(t)$ funksiyalar berilgan yetarlicha silliq funksiyalar.

Faraz qilaylik, ushbu maqola davomida berilgan $\varphi_1,\ \varphi_2,\ f,\ w$ va h funksiyalar quyidagi shartlar o‘rinli bo‘lsin:

B1) $w(x) \in C^2[0,1]$ va $w(0) = w(1) = 0$ va $w'(0) = w'(1)$;

B2) $\left(D_{0+,t}^{\alpha,\beta}h\right)(t) \in C[0,T]$, $|h(t)| \ge h_0 > 0$, $h_0 -$ berilgan son,

$$\int_0^1 w(x)\varphi_1(x)dx = I_{0+,t}^{(2-\alpha)(1-\beta)}h(t)_{t=0+}, \int_0^1 w(x)\varphi_2(x)dx$$
$$= \frac{\partial}{\partial t}\left(I_{0+,t}^{(2-\alpha)(1-\beta)}h\right)(t)_{t=0+}.$$

(11) tenglamani $w(x)$ ga ko‘paytiramiz va x bo‘yicha 0 dan 1 gacha integrallaymiz. Natijada biz quyidagiga ega bo‘lamiz

$$\int_0^1 w(x)\left\{\left(D_{0+,t}^{\alpha,\beta}u\right)(x,t) - u_{xx} + q(t)u(x,t)\right\}dx = \int_0^1 w(x)f(x,t)dx.$$

Yuqoridagi tenglikda qavslarni ochib, ko‘rinishini o‘zgartiramiz:

$$\int_0^1 w(x)\left(D_{0+,t}^{\alpha,\beta}u\right)(x,t)dx - \int_0^1 w(x)u_{xx}(x,t)dx$$
$$+\int_0^1 w(x)q(t)u(x,t)dx = \int_0^1 w(x)f(x,t)dx. \tag{24}$$

(3) chegaraviy shart va B1) shartga ko‘ra biz quyidagiga ega bo‘lamiz:

$$\int_0^1 w(x)u_{xx}(x,t)dx = \int_0^1 w''(x)\mathrm{u}(x,t)dx. \qquad (25)$$

(23) qo‘shimcha shart va (25) tenglikga ko‘ra, (24) ifodani ko‘rinishini o‘zgartiramiz:

$$\left(D_{0+,t}^{\alpha,\beta}h\right)(t) + q(t)h(t) - \int_0^1 w''(x)u(x,t)dx = \int_0^1 w(x)f(x,t)dx.$$

Bundan $q(t)$ nomalum koeffitsiyentni topamiz:

$$q(t) = \frac{1}{h(t)}\left(\int_0^1 w(x)f(x,t)dx - \left(D_{0+,t}^{\alpha,\beta}h\right)(t)\right) +$$

$$+\frac{1}{h(t)}\int_0^1 w''(x)\sum_{n=0}^{\infty} u_n(t)X_n(x)dx. \qquad (26)$$

$u_n(t)$ funksiya aniqlanishiga ko‘ra $q(t)$ koeffitsiyentga bog‘liq, ya’ni $u_n(t;q)$.

Oddiy almashtirishlardan so‘ng, q(t) koeffitsiyentni aniqlash masalasini unga ekvivalent quyidagi integral tenglamaga keltiramiz:

$$q(t) = q_0(t) + \frac{1}{h(t)}\sum_{n=1}^{\infty} w_n u_n(t;q), \qquad (27)$$

bu yerda

$$q_0(t) = \frac{1}{h(t)}\left(\int_0^1 w(x)f(x,t)dx - \left(D_{0+,t}^{\alpha,\beta}h\right)(t)\right),$$

$$w_n = \int_0^1 w''(x)X_n(x)dx,$$

$u_n(t)$ funksiya (13) munosabat orqali aniqlanadi.

(27) tenglikning o‘ng tomoni bilan aniqlanadigan F operatorni kiritamiz:

$$F[q](t) = q_0(t) + \frac{1}{h(t)}\sum_{n=1}^{\infty} w_n u_n(t;q).$$

So‘ngra (27) tenglamani operator tenglama ko‘rinishda yozib olamiz:

$$F[q](t) = q(t). \qquad (28)$$

Operator tenglamaning ozod hadi $q_0(t)$ ni baholab olamiz:

$$q_{00}:=\max_{t\in[0;T]}|q_0(t)|=\left\|\frac{1}{h(t)}\left(\int_0^1 w\,(x)f(x,t)dx-\left(D_{0+,t}^{\alpha,\beta}h\right)(t)\right)\right\|_{C[0,T]}.$$

$\rho>0$ tayinlaymiz (o'zgarmas) va Markazi $q_0(t)$ da radiusi ρ bo'lgan sharni kiritamiz:$B(q_0,\rho):=\{q(t):q(t)\in C[0,T],\ \| q-q_0\|\le\rho\}$.

Teskari masala uchun quyidagi teorema o'rinli:

2-Teorema. Faraz qilaylik, 1 teorema shartlari bajarilsin, bundan tashqari B1)-B2) shartlar bajarilsin. U holda shunday $T^*\in(0;T)$ soni mavjud bo'lib, (1)-(3), (23) teskari masalaning $q(t)\in C[0,T^*]$ sinfga tegishli yagona yechimi mavjud bo'ladi.

Isbot. Teskari masala uchun ushbu teoemani isbotlashda siqib akslantirish prinsipidan foydalanamiz. Buning uchun, birinchidan, yetarlicha kichik $T>0$ uchun F operatori $B(q_0,\rho)$ sharni o'ziga akslantirishini tekshiramiz, ya'ni $\forall q\in B(q_0,\rho)$ uchun $F[q](t)\in B(q_0,\rho)$ bo'lishini isbotlaylik. Haqiqatan ham, har qanday uzluksiz funksiya $q(t)$ uchun (27) formulaning o'ng yordamida aniqlangan $F[q](t)$ funksiya uzluksiz bo'ladi. Birinchi shartni tekshirish uchun quyidagi ayirmaning normasini baholab, biz buni topamiz:

$$\|\mathrm{F[q](t)-q_0(t)}\|=\left|\frac{1}{h(t)}\sum_{n=1}^{\infty}w_n\,u_n(T;q)\right|$$

$$\le\frac{w_0}{h_0}\sum_{n=1}^{\infty}\left(\frac{T^{\gamma+(\beta-1)(1-\alpha)}|\varphi_n|}{\Gamma\left(1+(\beta-1)(1-\alpha)\right)}+\frac{\|f_n\|_{C_\gamma[0,T]}\,T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)$$

$$\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}\,T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right).$$

Oxirgi tengsizlikning o'ng tomondagi qatorning tekis yaqinlashuvchanligini tekshiramiz. Buning uchun tengsizlikni quyidagicha yozamiz:

$$\|\mathrm{F[q](t)-q_0(t)}\|\le\frac{w_0}{h_0}\left(\frac{T^{\gamma+(\beta-1)(1-\alpha)}}{\Gamma\left(1+(\beta-1)(1-\alpha)\right)}\sum_{n=1}^{\infty}|\varphi_n|+\right.$$

$$\left.+\frac{T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}\|f_n\|_{C_\gamma[0,T]}\right)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}\,T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right).$$

3-Lemmaga ko'ra

$$\|\mathrm{F[q](t)-q_0(t)}\|\le\frac{w_0}{h_0}\left(\frac{T^{\gamma+(\beta-1)(1-\alpha)}}{\Gamma\left(1+(\beta-1)(1-\alpha)\right)}\sum_{n=1}^{\infty}\frac{|\varphi'_n|}{\lambda_n}+\right.$$

$$+\frac{T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}\frac{\| f_{nx} \|_{C_{\gamma}[0,T]}}{\lambda_n}\Bigg)E_{\alpha,\gamma}\left(\left(\| q \|_{C[0,T]}\, T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right). \qquad (29)$$

Bundan esa quyidagi tengsizlikga ega bo‘lamiz

$$\|\mathrm{F}[\mathrm{q}](\mathrm{t})-\mathrm{q}_0(\mathrm{t})\| \le \frac{Cw_0}{h_0}\left(\frac{T^{\gamma+(\beta-1)(1-\alpha)}\,\| \varphi' \|_{L_2[0,l]}}{\Gamma\big(1+(\beta-1)(1-\alpha)\big)}+\right.$$
$$\left.+\frac{T^{\alpha}B(\alpha,1-\gamma)\,\| f_x \|_{L_2[0,\mathrm{l}]}}{\Gamma(\alpha+1)}\right)E_{\alpha,\gamma}\left(\left(\| q \|_{C[0,T]}\, T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right).$$

E’tibor bering, bu tengsizlikda o‘ng tomonda hosil bo‘lgan funksiya T ga bog‘liq ortib boruvchi monoton bo‘lib, $q(t)$ funksiyaning $B(q_0,\rho)$ sharga tegishli ekanligi $\| \, \|q\|_{C[0,T]} \le \|q_0\|_{C[0,T]}+\rho$ tengsizlikni qanoatlantirishini bildiradi. Demak, bu tengsizlikdagi $\|q\|_{C[0,T]}$ ni $\|q_0\|_{C[0,T]}+\rho$ ifoda bilan almashtirsakgina tengsizlikni kuchaytiramiz. Ushbu almashtirishlarni amalga oshirib, biz quyidagi bahoni olamiz

$$\|\mathrm{F}[\mathrm{q}](\mathrm{t})-\mathrm{q}_0(\mathrm{t})\| \le \frac{Cw_0}{h_0}\left(\frac{T^{\gamma+(\beta-1)(1-\alpha)}\,\| \varphi' \|_{L_2[0,l]}}{\Gamma\big(1+(\beta-1)(1-\alpha)\big)}+\right.$$
$$\left.+\frac{T^{\alpha}B(\alpha,1-\gamma)\,\| f_x \|_{L_2[0,\mathrm{l}]}}{\Gamma(\alpha+1)}\right)E_{\alpha,\gamma}\left(\left(\left(\|q_0\|_{C[0,T]}+\rho\right)T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)$$

Faraz qilaylik, quyidagi T ga nisbatan tenglamaning musbat ildizi T_1 bo‘lsin. Shuning uchun agar T_1 bilan tenglamaning musbat ildizini belgilaymiz (T uchun)

$$\frac{Cw_0}{h_0}\left(\frac{T^{\gamma+(\beta-1)(1-\alpha)}\,\| \varphi' \|_{L_2[0,l]}}{\Gamma\big(1+(\beta-1)(1-\alpha)\big)}+\right.$$
$$\left.+\frac{T^{\alpha}B(\alpha,1-\gamma)\,\| f_x \|_{L_2[0,\mathrm{l}]}}{\Gamma(\alpha+1)}\right)E_{\alpha,\gamma}\left(\left(\left(\|q_0\|_{C[0,T]}+\rho\right)T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)=\rho,$$

u holda $T \le T_1$ uchun $\| F[q](t)-q_0(t) \| \le \rho$ tengsizlik bajariladi, yani $F[q](t) \in B(q_0,\rho)$ bo‘ladi. Demak, F operator $B(q_0,\rho)$ sharni o‘ziga akslantirdi.

Siqib akslantirishning ikkinchi sharti: $\forall q,\tilde{q} \in B(q_0,\rho)$ uchun

$$\| F[q](t)-F[\tilde{q}](t) \| \le \mathrm{N}\, \| q-\tilde{q} \|_{C[0,T]}$$

ekanligini tekshiramiz, bu yerda $0<\mathrm{N}<1$.

Avval biz, (13) integral tenglamaga mos $\tilde{u}_n$ yechimni quyidagi ko‘rinishda yozib olamiz

$$\tilde{u}_n(t)=\frac{t^{(\beta-1)(2-\alpha)}}{\Gamma\big(1+(\beta-1)(2-\alpha)\big)}\tilde{\varphi}_n+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tilde{f}_n(\tau)d\tau$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\tilde{q}(\tau)\tilde{u}_n(\tau)d\tau. \qquad (30)$$

(13) va (30) integral tenglamalardan $u_n(t)-\tilde{u}_n(t)=\bar{u}_n(t)$ ayirmani tuzamiz

$$\bar{u}_n(t)=\frac{t^{(\beta-1)(2-\alpha)}}{\Gamma\big(1+(\beta-1)(2-\alpha)\big)}\bar{\varphi}_n+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\bar{f}_n(\tau)d\tau$$
$$-\frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-\tau)^{\alpha-1}\,q(\tau)u_n(\tau)d\tau-\int_0^t (t-\tau)^{\alpha-1}\,\tilde{q}(\tau)\tilde{u}_n(\tau)d\tau\right).$$

Bunda quyidagidan foydalansak

$$q(\tau)u_n(\tau)-\tilde{q}(\tau)\tilde{u}_n(\tau)=u_n(\tau)\bar{q}(\tau)+\tilde{q}(\tau)\bar{u}_n(t),$$

quyidagini olamiz

$$\bar{u}_n(t)=\frac{t^{(\beta-1)(2-\alpha)}}{\Gamma\big(1+(\beta-1)(2-\alpha)\big)}\bar{\varphi}_n+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\bar{f}_n(\tau)d\tau$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,u_n(\tau)\bar{q}(\tau)d\tau-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\tilde{q}(\tau)\bar{u}_n(\tau)d\tau. \qquad (31)$$

$u_n-\tilde{u}_n$ ayirmaning $C_\gamma^{\alpha,\beta}[0,T]$ dagi norma bo‘yicha bahosini olamiz.

$$t^\gamma|\bar{u}_n|\le\left(\frac{t^{\gamma+(\beta-1)(2-\alpha)}|\bar{\varphi}_n|}{\Gamma\big(1+(\beta-1)(2-\alpha)\big)}+\frac{\|\bar{f}_n\|_{C_\gamma[0,T]}\,t^\alpha B(\alpha,1-\gamma)}{\alpha\Gamma(\alpha)}\right)$$
$$+\frac{\|\bar{q}\|_{C[0,T]}B(\alpha,1-\gamma)}{\Gamma(\alpha)}\left(\frac{t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_n|}{\Gamma\big(1+(\beta-1)(2-\alpha)\big)}\right.$$
$$\left.+\frac{\|f_n\|_{C_\gamma[0,T]}\,t^\alpha B(\alpha,1-\gamma)}{\alpha\Gamma(\alpha)}\right)$$
$$\times\,E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}B(\alpha,1-\gamma)t^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}t\right)$$
$$+\frac{\|\tilde{q}\|_{C[0,T]}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,|\bar{u}_n(\tau)|d\tau.$$

Oxirgi tengsizlikda Gronoul integral tengsizligini qo‘llasak, quyidagi tengsizlikni olamiz

$$t^\gamma|\bar{u}_n|\le\left[\left(\frac{t^{\gamma+(\beta-1)(2-\alpha)}|\bar{\varphi}_n|}{\Gamma\big(1+(\beta-1)(2-\alpha)\big)}+\frac{\|\bar{f}_n\|_{C_\gamma[0,T]}\,t^\alpha B(\alpha,1-\gamma)}{\alpha\Gamma(\alpha)}\right)\right.$$

$$+\frac{\|\tilde{q}\|_{C[0,T]}B(\alpha,1-\gamma)}{\Gamma(\alpha)}\left(\frac{t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_n|}{\Gamma\left(1+(\beta-1)(2-\alpha)\right)}\right.$$

$$\left.+\frac{\| f_n \|_{C_\gamma[0,T]}\, t^{\alpha}B(\alpha,1-\gamma)}{\alpha\Gamma(\alpha)}\right)$$

$$\left.\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}B(\alpha,1-\gamma)t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}t\right)\right]$$

$$E_{\alpha,\gamma}\left(\left(\|\tilde{q}\|_{C[0,T]}B(\alpha,1-\gamma)\right)^{\frac{1}{\alpha+\gamma-1}}t\right). \qquad (230)$$

Endi, ixtiyoriy $q(t),\ \tilde{q}(t)\in B(q_0,\rho)$ funksiyalarni olib, $C[0,T]$ fazoda ularning $F[q](t)$ va $F[\tilde{q}](t)$ tasvirlari orasidagi masofani hisoblaymiz. $\tilde{q}(t)$ ga mos keladigan $\tilde{u}_n(t)$ funksiyasi $\varphi_n=\tilde{\varphi}_n,\ \psi_n=\tilde{\psi}_n$ va $f_n=\tilde{f}_n$ bilan (13), (27), (31) integral tenglamalarni qanoatlantiradi. (13), (27), (31) tenglamalar yordamida $F[q](t)-F[\tilde{q}](t)$ ayirmasini tuzib, keyin uning normasini baholaymiz:

$$\| F[q](t)-F[\tilde{q}](t) \|\leq \frac{w_0}{h_0^2}\frac{B(\alpha,1-\gamma)}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\left[\frac{t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_n|}{\Gamma\left(1+(\beta-1)(2-\alpha)\right)}+\right.$$

$$\left.\frac{\| f_n \|_{C_\gamma[0,T]}\, t^{\alpha}B(\alpha,1-\gamma)}{\alpha\Gamma(\alpha)}\right]E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}B(\alpha,1-\gamma)t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}t\right)$$

$$\times E_{\alpha,\gamma}\left(\left(\|\tilde{q}\|_{C[0,T]}B(\alpha,1-\gamma)\right)^{\frac{1}{\alpha+\gamma-1}}t\right)\|q-\tilde{q}\|_{C[0,T]}. \qquad (231)$$

(t) va $\tilde{q}(t)$ funksiyalar $B(q_0,\rho)$ sharga tegishli, bu funksiyalarning har biri uchun $\|q\|_{C[0,T]}\leq\|q_0\|_{C[0,T]}+\rho$ tengsizlikka ega. E’tibor bering, $\| q-\tilde{q} \|_{C[0,T]}$ koeffitsientidagi tengsizlikda (231) o‘ng tomondagi funksiya $\| q \|$, $\| \tilde{q} \|$ va T bilan ortib boruvchi monotondir. Demak, ‖q‖ va $\|\tilde{q}\|$ o‘rniga $\|q_0\|_{C[0,T]}+\rho$ qo‘yiladi. tengsizlikda (231) bilan $\|q_0\|_{C[0,T]}+\rho$ faqat tengsizlikni kuchaytiradi. Bundan biz ushbuga ega bo‘lamiz

$$\| F[q](t)-F[\tilde{q}](t) \|\leq \frac{w_0}{h_0^2}\frac{B(\alpha,1-\gamma)}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\left[\frac{t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_n|}{\Gamma\left(1+(\beta-1)(2-\alpha)\right)}+\right.$$

$$\left.\frac{\| f_n \|_{C_\gamma[0,T]}\, t^{\alpha}B(\alpha,1-\gamma)}{\alpha\Gamma(\alpha)}\right]E_{\alpha,\gamma}\left(\left(\left(\|q_0\|_{C[0,T]}+\rho\right)B(\alpha,1-\gamma)t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}t\right)$$

$$\times E_{\alpha,\gamma}\left(\left(\left(\|q_0\|_{C[0,T]}+\rho\right)B(\alpha,1-\gamma)\right)^{\frac{1}{\alpha+\gamma-1}}t\right)\|q-\tilde{q}\|_{C[0,T]}.$$

Shunday qilib, agar T_2 −quyidagi tenglamaning musbat ildizi bo‘lsa (T uchun)

$$\frac{w_0}{h_0^2}\frac{B(\alpha,1-\gamma)}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\left[\frac{t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_n|}{\Gamma\left(1+(\beta-1)(2-\alpha)\right)}+\right.$$

$$\left.\frac{\| f_n \|_{C_\gamma[0,T]}\, t^{\alpha}B(\alpha,1-\gamma)}{\alpha\Gamma(\alpha)}\right]E_{\alpha,\gamma}\left(\left(\left(\|q_0\|_{C[0,T]}+\rho\right)B(\alpha,1-\gamma)t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}t\right)$$

$$\times E_{\alpha,\gamma}\left(\left(\left(\|q_0\|_{C[0,T]}+\rho\right)B(\alpha,1-\gamma)\right)^{\frac{1}{\alpha+\gamma-1}}t\right)=1,$$

u holda $T\in(0,T_2)$ uchun F operatori $q(t)$, $\tilde{q}(t)\in B(q_0,\rho)$ elementlari orasidagi masofani qisqartirib akslantiradi.

Shunday qilib, siquvchan akslantirishlar prinsipining ikkila shartini tekshirdik. Binobarin, agar $T^*<min(T_1,T_2)$ ni tanlasak, F operatori $B(q_0,\rho)$ shardagi qisqarish akslantirishdir. Bundan, Banax teoremasiga ko‘ra, F operatori $B(q_0,\rho)$ sharida yagona qo‘zg‘almas nuqtaga ega, ya’ni tenglamaning yagona yechimi mavjud ([26]). 23-teorema isbotlandi.

Xulosa

Ushbu maqolada kasr tartibli umumlashgan Riman Liuvill differensial operatorli tenglama uchun to‘g‘ri hamda teskari masala qaralgan. Boshlang‘ich-nolokal chegaraviy shartli masalani tadqiq etishda, dastlab fazo o‘zgaruvchisiga bog‘liq spektral masala o‘rganildi. Spektral masalaning xos soni va xos funksiyalari aniqlandi. Xos funksiyalari qaralayotgan fazoda to‘la bazis tashkil etganligidan boshlang‘ich chegaraviy masalaning yechimi shu bazis orqali qatorga yoyilgan holda qidirildi. Vaqt o‘zgaruvchisi bo‘yicha Koshi masalasi olindi. Bu Koshi masalasi ekvivalent bo‘lgan integral tenglama olindi. Integral tenglamaning yechimi mavjudligi va yagonaligi isbotlandi. So‘ngra boshlang‘ich chegaraviy masala yechimi qator ko‘rinishda izlaydi. Qatorning tekis darajada uzluksizligi isbotlaydi. Shundan so‘ng kasr tartibli umumlashgan Riman Liuvill differensial operatorli tenglama uchun teskari masala” tadqiq etildi, unda to‘g‘ri masala yechimiga qo‘shimcha shart berish orqali tenglamada qatnashuvchi nomalum koeffitsiyentni aniqlash masalasi o‘rganildi. Bunda asosan teskari masala unga ekvivalent integral tenglamaga keltirildi, so‘ngra siqiluvchan akslantirishlar prinsipi yordamida teskari masalaning yechimi mavjudligi va yagonaligi isbotlandi.

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